

# Fokker - Planck equation for incompressible fluid

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## Abstract

In this article we derive Fokker - Planck equation for incompressible fluid and investigate its properties

## Keywords

Fokker-Planck equation, continuum mechanics, incompressible fluid

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## 1 Introduction

The object of our considerations is a special case of Fokker - Planck equation, which describes evolution of 3D continuum of non-interacting particles imbedded in a dense medium without outer forces. The interaction between particles and medium combines diffusion in physical space and velocities space. Classic Fokker - Planck equation contains only two forces, which act on particles: damping force proportional to velocity and random force.

The aim of this work is to derive equation for the case of constrained movement - the space density of particles remains constant. As a result, our equation contains the third force - constraint reaction. This is equivalent to hydrostatic pressure.

Our starting point is the classical Fokker - Planck equation:

$$\frac{\partial n}{\partial t} + v_k \frac{\partial n}{\partial x_k} - \alpha \frac{\partial}{\partial v_j} (v_j n) = k \frac{\partial^2 n}{\partial v_j \partial v_j}. \quad (1)$$

where

$n = n(t, x_1, x_2, x_3, v_1, v_2, v_3)$  - density;

$t$  - time variable;

$x_1, x_2, x_3$  - space coordinates;

$v_1, v_2, v_3$  - velocities;

$\alpha$  - coefficient of damping;

$k$  - coefficient of diffusion.

In the following section we add to this equation terms due to reaction of constraints and thus derive the Fokker - Planck equation for incompressible fluid. Then we rewrite the equation for curvilinear coordinates. In the fourth section we obtain a stationary solution of Fokker - Planck equation for incompressible fluid and in the fifth section we derive linearized equation. In the last section we introduce symmetries of Fokker - Planck equation for incompressible fluid.

## 2 Derivation of Fokker - Planck equation for incompressible fluid

In our previous work (see ref. [1-2]) we deducted equation (1) from the kinematic equation (see below). The crucial point is expression for the force, which acts upon particles. We represented the force as a sum of random force  $R_i$  and friction force, which is proportional to the velocity:

$$Q_i = R_i - \alpha v_i. \quad (2)$$

We calculated random force  $R_i$  from the law

$$nR_i = -k \frac{\partial n}{\partial v^i}. \quad (3)$$

(2-3) give explicit expressions for the forces. As we show in [2], (2-3) imply equation (1).

Now we try to get expression for reaction of constraint. We need some additional principle for this purpose. In the following we accept the Gauss principle of least constraint (see [3]).

Gauss principle reads for discrete system of particles:

$$G = \frac{1}{2} \sum m_r \left\{ \left( \ddot{x}_r - \frac{X_r}{m_r} \right)^2 + \left( \ddot{y}_r - \frac{Y_r}{m_r} \right)^2 + \left( \ddot{z}_r - \frac{Z_r}{m_r} \right)^2 \right\} \quad (4)$$

where

$x_r, y_r, z_r$  - material point coordinates ;

$\ddot{x}_r, \ddot{y}_r, \ddot{z}_r$  - accelerations ;

$X_r, Y_r, Z_r$  - components of outer forces, acting on this point. They are dependent on coordinates  $x_r, y_r, z_r$  and velocities  $\dot{x}_r, \dot{y}_r, \dot{z}_r$ .

Besides  $X_r, Y_r, Z_r$  reactions of constraints are also to be considered, though they are not given explicitly. They appear as Lagrange multipliers of constraints.

Let us suppose, that coordinates  $x_r, y_r, z_r$  and velocities  $\dot{x}_r, \dot{y}_r, \dot{z}_r$  are kinematically possible, that is they satisfy constraints (holonomic or nonholonomic). Then effective accelerations  $\ddot{x}_r, \ddot{y}_r, \ddot{z}_r$  provide minimum of  $G$ , comparing with all kinematically possible accelerations.

The derivation of reactions of constrains for the simple case of discrete system of particles is straightforward (see [3]). We deal with more complicated case of continuum with dispersed velocities - many particles with different velocities are simultaneously placed in each point of space.

But first of all we examine first intermediate case of continuum with uniquely defined velocities field, which is simpler. For this case the Gauss principle was used in [4], [5].

For such a continuum (fluid) we have the following expression for accelerations:

$$a_i = \frac{\partial u_i}{\partial t} + u_k \frac{\partial u_i}{\partial x_k}. \quad (5)$$

In this way we get the following expression for Gauss functional:

$$G = \int \frac{\rho}{2} \left( a_m - \frac{1}{\rho} F_m \right) \left( a_m - \frac{1}{\rho} F_m \right) dV. \quad (6)$$

Condition of incompressibility of the velocities field  $u_i$  is

$$\frac{\partial u_k}{\partial x_k} = 0. \quad (7)$$

Differentiating (7) on time, we get expression for constraint on accelerations field:

$$\frac{\partial^2 u_k}{\partial x_k \partial t} = 0. \quad (8)$$

This implies the following constraint on accelerations  $a^i$ :

$$\frac{\partial a_k}{\partial x_k} = \frac{\partial \dot{u}_k}{\partial x_k} + \frac{\partial}{\partial x_m} \left( u_k \frac{\partial u_m}{\partial x_k} \right) = \frac{\partial u_k}{\partial x_m} \frac{\partial u_m}{\partial x_k}. \quad (9)$$

To take in account (9) we use Lagrange multipliers method. This gives following modified expression for Gauss functional:

$$G = \int \left[ \frac{\rho}{2} \left( a_m - \frac{1}{\rho} F_m \right) \left( a_m - \frac{1}{\rho} F_m \right) - p \left( \frac{\partial a_k}{\partial x_k} - \frac{\partial u_k}{\partial x_m} \frac{\partial u_m}{\partial x_k} \right) \right] dV. \quad (10)$$

where  $p$  is Lagrange multiplier for constraint (9). We seek the extreme of functional (10) with  $a_j$  as unknown variables.

Variational Euler equations for functional (10) are

$$\rho \left( \dot{u}_m + u_i \frac{\partial u_m}{\partial x_i} - \frac{1}{\rho} F_m \right) + \frac{\partial p}{\partial x_m} = 0. \quad (11)$$

They are identical to standard Euler equations for incompressible fluid. We see, that Lagrange's multiplier  $p$  is pressure and  $1/\rho (\partial p / \partial x^k)$  is expression for reaction of incompressibility constraint.

We conclude, that as for get Fokker - Planck equation for incompressible fluid we should:

- to add to system additional equation - expression for incompressibility constraint;

- to add to all forces, acting on particle (see (2)), gradient of the pressure as reaction of ideal constraint .

$$Q_i = R_i - \frac{1}{\rho} \frac{\partial p}{\partial q^i} - \alpha v_i. \quad (12)$$

In this way we get the following system with one integral constraint expression and one differential equation.

$$\rho = \int_V n \, dv_1 dv_2 dv_3 = \text{const}, \quad (13)$$

where  $\rho$  is constant density.

$$\frac{\partial n}{\partial t} + v_k \frac{\partial n}{\partial x_k} - \alpha \frac{\partial(v_j n)}{\partial v_j} - \frac{1}{\rho} \frac{\partial n}{\partial v_j} \frac{\partial p}{\partial x_j} = k \frac{\partial^2 n}{\partial v_j \partial v_j}. \quad (14)$$

(13) imply equations  $(\partial \rho / \partial x_i) = 0$ .

This derivation method is insufficient, because we obtained expression (12) from the wrong model. Now we return to the case of continuum with dispersed velocities - there exist many particles with different velocities in each point of space.

Kinematic of single particle is described by differential equations:

$$\frac{dx_k}{dt} = v_k; \quad (15)$$

$$\frac{dv_k}{dt} = b_k; \quad (16)$$

where

$x_k$  - space coordinates;

$v_k$  - velocities of particles;

$b_k$  - accelerations of particles.

Evolution of particles density  $n$  satisfies the following equation:

$$\frac{\partial n}{\partial t} + \frac{\partial(nv_k)}{\partial x_k} + \frac{\partial(nb_k)}{\partial v_k} = 0. \quad (17)$$

If accelerations are independent from  $n$ , (15) and (16) are equations of characteristics of (17).

The dynamics of the system is equation of movement of particles - the second Newton's law:

$$b_j = \frac{1}{m} Q_j; \quad (18)$$

where  $m$  - particles mass, the force  $Q_j$  is the same as in (2-3), (12).

We accept (13) as constraint on  $n$  and try to get expression for constraints on accelerations  $b_j$ . (13) implies

$$\int_V \frac{\partial n}{\partial t} dv_1 dv_2 dv_3 = 0, \quad (19)$$

$$\int_V \frac{\partial n}{\partial x_j} dv_1 dv_2 dv_3 = 0, \quad (20)$$

This means, that averages of  $n$  derivatives on space and time coordinates are zero.

Let us integrate (17) on velocities and use (20). We get identity:

$$\int_V \frac{\partial(nv_k)}{\partial x_k} dv_1 dv_2 dv_3 = 0. \quad (21)$$

But we know, that average velocity  $u_k$  is equal to

$$\rho u_k = \int_V nv_k dv_1 dv_2 dv_3. \quad (22)$$

Therefore (22) is similar to (7) - the divergence of average velocity field is zero. We can write this in the following way:

$$\int_V \frac{\partial n}{\partial x_k} v_k dv_1 dv_2 dv_3 = 0. \quad (23)$$

This identity concerns the first moments of derivatives of  $n$  on space coordinates. They are not zero, but their sum is zero.

Let us multiply (17) by  $v_j$  and integrate the result by parts:

$$\frac{\partial(nv_j)}{\partial t} + \frac{\partial(nv_k v_j)}{\partial x_k} + \frac{\partial(nb_k v_j)}{\partial v_k} = nb_j. \quad (24)$$

Let us integrate (25) on velocities. We get expression for average accelerations (compare this with (5))

$$\rho(b_j)_{avg} = \int_V nb_j dv_1 dv_2 dv_3 = \int_V \left( \frac{\partial(nv_j)}{\partial t} + \frac{\partial(nv_k v_j)}{\partial x_k} \right) dv_1 dv_2 dv_3. \quad (25)$$

Let us denote

$$J_{kj} = \int_V nv_k v_j dv_1 dv_2 dv_3 = \rho u_k u_j - \sigma_{kj}, \quad (26)$$

where  $\sigma_{kj}$  - components of stresses tensor.

So (26) reads

$$\rho(b_j)_{avg} = \frac{\partial(\rho u_j)}{\partial t} + \frac{\partial(\rho u_k u_j)}{\partial x_k} - \frac{\partial \sigma_{kj}}{\partial x_k} = \rho a_j - \frac{\partial \sigma_{kj}}{\partial x_k}. \quad (27)$$

This is very remarkable, that average acceleration is not equal to acceleration of average motion and contain additional term - divergence of stresses tensor. This follows from the fact, that accelerations are quadratic on velocities.

We shall use (27) to get constraint expression.

Integrate Newton's law (18) and get:

$$\rho(b_j)_{avg} = (Q_j)_{avg} = F_j. \quad (28)$$

(27) and (28) imply equations of movement :

$$\rho a_j - \frac{\partial \sigma_{kj}}{\partial x_k} = F_j. \quad (29)$$

The Gauss functional without constraints has the following form

$$G = \int_X dx_1 dx_2 dx_3 \int_V \frac{n}{2} \left( b_k - \frac{1}{m} Q_k \right) \left( b_k - \frac{1}{m} Q_k \right) dv_1 dv_2 dv_3. \quad (30)$$

To get expression for constraint on accelerations we take divergence of (27) (compare with (9)).

$$\int_V \frac{\partial}{\partial x_j} (nb_j) dv_1 dv_2 dv_3 = \int_V \frac{\partial}{\partial x_j} \left( \frac{\partial(nv_j)}{\partial t} + \frac{\partial(nv_k v_j)}{\partial x_k} \right) dv_1 dv_2 dv_3. \quad (31)$$

The first term in the RHS of (31) is zero according to (21). The rest terms give us the desired constraint expression:

$$\int_V \left[ \frac{\partial}{\partial x_j} (nb_j) - \frac{\partial^2 n}{\partial x_j \partial x_k} v_k v_j \right] dv_1 dv_2 dv_3 = 0. \quad (32)$$

We multiply (32) by  $p/\rho$  and add the result to (31). Gauss functional with constraint reads now (compare with (10)):

$$G = \int_X dx_1 dx_2 dx_3 \int_V \left\{ \frac{n}{2} \left( b_k - \frac{1}{m} Q_k \right) \left( b_k - \frac{1}{m} Q_k \right) - \frac{p(x_i)}{\rho} \left[ \frac{\partial}{\partial x_j} (nb_j) - \frac{\partial^2 n}{\partial x_j \partial x_k} v_k v_j \right] \right\} dv_1 dv_2 dv_3. \quad (33)$$

We seek the extreme of functional (33) with  $(nb_j)$  as unknown variables. The Euler's equation for (33) is:

$$nb_k = \frac{n}{m} Q_k - \frac{1}{\rho} n \frac{\partial p}{\partial x_k}. \quad (34)$$

This means, that reaction of constrains is really equivalent to pressure field and we return to (12) and (14) once again.

### 3 Fokker - Planck equation for incompressible fluid in curvilinear coordinates

Now let us consider the form of (13-14) system in the curvilinear coordinates. We need not perform all calculations here, because we can refer for details to our previous works [1-2]. Therefore we give here only results of calculations.

Let us denote  $g_{mn}$  - the covariant components of metric tensor and  $g^{mn}$  - the contravariant components of metric tensor,  $g = \det|g_{ij}| = 1/\det|g^{ij}|$ .

Christoffel's symbol is called

$$\Gamma_{p,mn} = \frac{1}{2} \left( \frac{\partial g_{np}}{\partial x^m} + \frac{\partial g_{pm}}{\partial x^n} - \frac{\partial g_{mn}}{\partial x^p} \right). \quad (35)$$

and

$$\Gamma_{mn}^p = g^{pq} \Gamma_{q,mn}. \quad (36)$$

Covariant components of velocity vector are  $v_i$ , contravariant components of velocity vector are  $v^i$ .

Using these definitions, we can write equations (13-14) in curvilinear coordinates with contravariant velocities as independent variables, as

$$\int_V n \sqrt{g} dv^1 dv^2 dv^3 = 1. \quad (37)$$

$$\begin{aligned} \frac{\partial n}{\partial t} + v^k \frac{\partial n}{\partial x^k} - \Gamma_{pq}^k v^p v^q \frac{\partial n}{\partial v^k} - \alpha v^k \frac{\partial n}{\partial v^k} - 3 \alpha n - \\ - \frac{1}{\rho} g^{mn} \frac{\partial n}{\partial v^m} \frac{\partial p}{\partial x^n} = k g^{lk} \frac{\partial^2 n}{\partial v^l \partial v^k}. \end{aligned} \quad (38)$$

For details we refer to our works [1-2].

For example, the system (37-38) in spherical coordinates reads:

$$\int_V n dv^1 dv^2 dv^3 = \frac{\rho}{r^2 \sin(\theta)}. \quad (39)$$

$$\begin{aligned} \frac{\partial n}{\partial t} + v^1 \frac{\partial n}{\partial r} + v^2 \frac{\partial n}{\partial \theta} + v^3 \frac{\partial n}{\partial \phi} + \\ + r \left( v^2 v^2 + \sin^2(\theta) v^3 v^3 \right) \frac{\partial n}{\partial v^1} + \left( \sin(\theta) \cos(\theta) v^3 v^3 - \frac{2}{r} v^1 v^2 \right) \frac{\partial n}{\partial v^2} - 2 \left( v^1 v^3 + \frac{\cos(\theta)}{\sin(\theta)} v^2 v^3 \right) \frac{\partial n}{\partial v^3} - \\ - \alpha \left( v^1 \frac{\partial n}{\partial v^1} + v^2 \frac{\partial n}{\partial v^2} + v^3 \frac{\partial n}{\partial v^3} \right) - 3 \alpha n - \frac{1}{\rho} \left( \frac{\partial n}{\partial v^1} \frac{\partial p}{\partial r} + \frac{1}{r^2} \frac{\partial n}{\partial v^2} \frac{\partial p}{\partial \theta} + \frac{1}{r^2 \sin^2(\theta)} \frac{\partial n}{\partial v^3} \frac{\partial p}{\partial \phi} \right) = \\ = k \left( \frac{\partial^2 n}{\partial v^1 \partial v^1} + \frac{1}{r^2} \frac{\partial^2 n}{\partial v^2 \partial v^2} + \frac{1}{r^2 \sin^2(\theta)} \frac{\partial^2 n}{\partial v^3 \partial v^3} \right). \end{aligned} \quad (40)$$

The system (13-14) in curvilinear coordinates and with covariant velocities as independent variables, is

$$\int_V n \frac{1}{\sqrt{g}} dv_1 dv_2 dv_3 = \rho. \quad (41)$$

$$\begin{aligned} \frac{\partial n}{\partial t} + g^{mk} v_m \frac{\partial n}{\partial x^k} + \Gamma_{kl}^q g^{pl} v_p v_q \frac{\partial n}{\partial v_k} - \alpha v_k \frac{\partial n}{\partial v_k} - 3 \alpha n - \\ - \frac{1}{\rho} \frac{\partial n}{\partial v_k} \frac{\partial p}{\partial x^k} = k g_{lk} \frac{\partial^2 n}{\partial v_l \partial v_k}. \end{aligned} \quad (42)$$

For example, the system (41-42) in spherical coordinates has the form:

$$\int_V n dv_1 dv_2 dv_3 = r^2 \sin(\theta) \rho. \quad (43)$$

$$\begin{aligned} \frac{\partial n}{\partial t} + v_1 \frac{\partial n}{\partial r} + \frac{v_2}{r^2} \frac{\partial n}{\partial \theta} + \frac{v_3}{r^2 \sin^2(\theta)} \frac{\partial n}{\partial \phi} + \\ + \frac{1}{r^3} \left( v_2 v_2 + \frac{v_3 v_3}{\sin^2(\theta)} \right) \frac{\partial n}{\partial v_1} + \frac{\cos(\theta) v_3 v_3}{r^2 \sin^3(\theta)} \frac{\partial n}{\partial v_2} - \\ - \alpha \left( v_1 \frac{\partial n}{\partial v_1} + v_2 \frac{\partial n}{\partial v_2} + v_3 \frac{\partial n}{\partial v_3} \right) - 3 \alpha n - \frac{1}{\rho} \left( \frac{\partial n}{\partial v^1} \frac{\partial p}{\partial r} + \frac{\partial n}{\partial v^2} \frac{\partial p}{\partial \theta} + \frac{\partial n}{\partial v^3} \frac{\partial p}{\partial \phi} \right) = \\ = k \left( \frac{\partial^2 n}{\partial v_1 \partial v_1} + r^2 \frac{\partial^2 n}{\partial v_2 \partial v_2} + r^2 \sin^2(\theta) \frac{\partial^2 n}{\partial v_3 \partial v_3} \right). \end{aligned} \quad (44)$$

For orthogonal coordinates the diagonal components of metric tensor are expressed as squares of  $H_i$  - Lamé coefficients. All off-diagonal components are zero. The Christoffel's symbol components can be expressed as derivatives of the Lamé coefficients.

The system (13-14) in curvilinear coordinates and with physical velocities as independent variables, is

$$\int_V n dw^1 dw^2 dw^3 = \rho. \quad (45)$$

$$\begin{aligned} \frac{\partial n}{\partial t} + \frac{w^k}{H_k} \frac{\partial n}{\partial x^k} + \frac{\partial n}{\partial w^k} \frac{w^s}{H_s H_k} \left( w^s \frac{\partial H_s}{\partial x^k} - w^k \frac{\partial H_k}{\partial x^s} \right) - \alpha w^k \frac{\partial n}{\partial w^k} - 3 \alpha n - \\ - \frac{1}{\rho} \frac{1}{H_k} \frac{\partial n}{\partial w^k} \frac{\partial p}{\partial x^k} = k \frac{\partial^2 n}{\partial w^i \partial w^i}. \quad (s! = k) \end{aligned} \quad (46)$$

For example, the system (45-46) in spherical coordinates has the form:

$$\int_V n dw^1 dw^2 dw^3 = \rho. \quad (47)$$

$$\begin{aligned} \frac{\partial n}{\partial t} + w^1 \frac{\partial n}{\partial r} + \frac{w^2}{r} \frac{\partial n}{\partial \theta} + \frac{w^3}{r \sin(\theta)} \frac{\partial n}{\partial \phi} + \\ + \frac{1}{r} \left( w^2 w^2 + w^3 w^3 \right) \frac{\partial n}{\partial w^1} + \frac{1}{r} \left( \frac{\cos(\theta)}{\sin(\theta)} w^3 w^3 - w^1 w^2 \right) \frac{\partial n}{\partial w^2} - \frac{1}{r} \left( w^1 w^3 + \frac{\cos(\theta)}{\sin(\theta)} w^2 w^3 \right) \frac{\partial n}{\partial w^3} - \end{aligned} \quad (48)$$



$$- \alpha \left( w^1 \frac{\partial n}{\partial w^1} + w^2 \frac{\partial n}{\partial w^2} + w^3 \frac{\partial n}{\partial w^3} \right) - 3 \alpha n -$$

$$- \frac{1}{\rho} \left( \frac{\partial n}{\partial v^1} \frac{\partial p}{\partial r} + \frac{1}{r} \frac{\partial n}{\partial v^2} \frac{\partial p}{\partial \theta} + \frac{1}{r \sin(\theta)} \frac{\partial n}{\partial v^3} \frac{\partial p}{\partial \phi} \right) = k \left( \frac{\partial^2 n}{\partial w^1 \partial w^1} + \frac{\partial^2 n}{\partial w^2 \partial w^2} + \frac{\partial^2 n}{\partial w^3 \partial w^3} \right).$$

## 4 Stationary solution of Fokker - Planck equation for incompressible fluid

Only in this section we discuss the case of nonzero force field. In this case equation (14) reads

$$\frac{\partial n}{\partial t} + v_j \frac{\partial n}{\partial x_j} - \alpha \frac{\partial}{\partial v_j} (v_j n) + \frac{1}{\rho} \frac{\partial n}{\partial v_j} \left( F_j - \frac{\partial p}{\partial x_j} \right) = k \frac{\partial^2 n}{\partial v_j \partial v_j}. \quad (49)$$

where  $F_i(\vec{x})$  are components of force acting on particle. For non potential forces the stationary solution of Fokker - Planck equation does not exist. For potential forces we have the following expressions for force components:

$$F_i = - \frac{\partial P_i}{\partial x_i}, \quad (50)$$

and (14) reads

$$\frac{\partial n}{\partial t} + v_j \frac{\partial n}{\partial x_j} - \alpha \frac{\partial}{\partial v_j} (v_j n) - \frac{\partial n}{\partial v_j} \frac{\partial}{\partial x_j} (P_i + p) = k \frac{\partial^2 n}{\partial v_j \partial v_j}; \quad (51)$$

where  $P_i(\vec{x})$  - potential function.

Stationary solution of usual Fokker - Planck equation has the form  $n = m(\vec{x}) s(\vec{v})$ . For the case of incompressible fluid must be  $m(\vec{x}) = 1$  because of condition (13). Therefore  $n = n(\vec{v})$ .

To kill all the terms with cross products of derivatives on space coordinates and velocities in (51), we substitute:

$$p = const - P_i. \quad (52)$$

This is the Pascal's law.

The rest of terms in (51), depending only on velocities, is the divergence of some current in velocities space. For the true static solution all components of the current must be zero:

$$-\alpha v_j n = k \frac{\partial n}{\partial v_j}. \quad (53)$$

So  $n$  has Maxwell distribution :

$$n = n_0 \exp \left[ - \frac{\alpha}{2k} v_j v_j \right]. \quad (54)$$

The value of  $n_0$  constant factor we find from condition (13)

$$n_0 = \rho \left( \frac{\alpha}{2\pi k} \right)^{3/2}. \quad (55)$$

$$n = \rho \left( \frac{\alpha}{2\pi k} \right)^{3/2} \exp \left[ -\frac{\alpha}{2k} v_j v_j \right]. \quad (56)$$

## 5 Linearization of Fokker - Planck equation for incompressible fluid

We seek solution of (13-14) as a sum of stationary solution and perturbation term:

$$p = \epsilon \bar{p}. \quad (57)$$

$$n = \rho \left( \frac{\alpha}{2\pi k} \right)^{3/2} \exp \left[ -\frac{\alpha}{2k} v_j v_j \right] + \epsilon \bar{n}. \quad (58)$$

( $Pi = 0$  here).

Substitute these expressions to (13-14) and drop terms of more than first degree on  $\epsilon$ .

$$\int_V \bar{n} dv_x dv_y dv_z = 0. \quad (59)$$

$$\frac{\partial \bar{n}}{\partial t} + v_j \frac{\partial \bar{n}}{\partial x_j} - \alpha \frac{\partial}{\partial v_j} (v_j \bar{n}) + \left( \frac{\alpha}{k} \right) \left( \frac{\alpha}{2\pi k} \right)^{3/2} \exp \left[ -\frac{\alpha}{2k} v_j v_j \right] v_k \frac{\partial \bar{p}}{\partial x_k} = k \frac{\partial^2 \bar{n}}{\partial v_j \partial v_j}. \quad (60)$$

The system (59-60) though linear, is not trivial. We shall discuss it's solution later.

## 6 Symmetries of Fokker - Planck equation for incompressible fluid

We presented symmetries of standard Fokker - Planck equation (1) in our previous work [6] with all calculations details. In this section we present only result of calculations for Fokker - Planck equation for incompressible fluid.

Two equations of the (13-14) have different nature: (13) is integral equation and (14) is differential. To obtain symmetries of this system we perform two-step process: 1) we determine symmetries of differential equation (14) and 2) we calculate actions of these symmetries operators upon (13) and drop all invalid operators, for which action does not vanish.

We omit tedious details of step 1) and give below only resulting expressions. Expressions for variations of variables are:

$$\delta x = r_2 z - r_3 y + 3 C_1 e^{\alpha/2t} x + f_1(t); \quad (61)$$

$$\begin{aligned}\delta y &= r_3 x - r_1 z + 3 C_1 e^{\alpha t/2} y + f_2(t); \\ \delta z &= r_1 y - r_2 x + 3 C_1 e^{\alpha t/2} z + f_3(t).\end{aligned}$$

where  $r_1, r_2, r_3, C_1$  - arbitrary constant coefficients;  $f_1, f_2, f_3$  - arbitrary functions of argument  $t$ .

$$\begin{aligned}\delta u &= \frac{3}{2} \alpha C_1 e^{\alpha t/2} x + C_1 e^{\alpha t/2} u + r_2 w - r_3 v + f_1'(t); \\ \delta v &= \frac{3}{2} \alpha C_1 e^{\alpha t/2} y + C_1 e^{\alpha t/2} v + r_3 u - r_1 w + f_2'(t); \\ \delta w &= \frac{3}{2} \alpha C_1 e^{\alpha t/2} z + C_1 e^{\alpha t/2} w + r_1 v - r_2 u + f_3'(t);\end{aligned}\tag{62}$$

$$\delta t = \frac{4}{\alpha} C_1 e^{\alpha t/2} + C_2.\tag{63}$$

$$\delta n = f_4(n e^{3\alpha t}) e^{-3\alpha t} - 12 C_1 e^{\alpha t/2} n.\tag{64}$$

$$\begin{aligned}\delta p &= (\alpha f_1' - f_1'')x + (\alpha f_2' - f_2'')y + (\alpha f_3' - f_3'')z + \\ &+ \frac{3}{8} \alpha^2 C_1 e^{\alpha t/2} (x^2 + y^2 + z^2) + 2 C_1 e^{\alpha t/2} p + f_5(t).\end{aligned}\tag{65}$$

where  $C_2$  - one more arbitrary constant coefficient;  $f_4, f_5$  - two more arbitrary functions of argument  $t$ .

These expressions lead to the following expressions for symmetries operators:

- operator (rather exotic) associated with  $C_1$ :

$$\begin{aligned}v_1 &= e^{\alpha t/2} \left[ -12n \frac{\partial}{\partial n} + \left( \frac{3}{8} \alpha^2 (x^2 + y^2 + z^2) + 2p \right) \frac{\partial}{\partial p} + \frac{4}{\alpha} \frac{\partial}{\partial t} + \right. \\ &\left. + \left( \frac{3}{2} \alpha x + u \right) \frac{\partial}{\partial u} + \left( \frac{3}{2} \alpha y + v \right) \frac{\partial}{\partial v} + \left( \frac{3}{2} \alpha z + w \right) \frac{\partial}{\partial w} + 3x \frac{\partial}{\partial x} + 3y \frac{\partial}{\partial y} + 3z \frac{\partial}{\partial z} \right];\end{aligned}\tag{66}$$

- time shift operator associated with  $C_2$ :

$$v_2 = \frac{\partial}{\partial t};\tag{67}$$

- three operators of shifts along  $x, y, z$  axes associated with  $f_1, f_2, f_3$  functions:

$$v_3 = (\alpha f_1' - f_1'')x \frac{\partial}{\partial p} + f_1'(t) \frac{\partial}{\partial u} + f_1(t) \frac{\partial}{\partial x};\tag{68}$$

$$v_4 = (\alpha f_2' - f_2'')y \frac{\partial}{\partial p} + f_2'(t) \frac{\partial}{\partial v} + f_2(t) \frac{\partial}{\partial y};\tag{69}$$

$$v_5 = (\alpha f_3' - f_3'')z \frac{\partial}{\partial p} + f_3'(t) \frac{\partial}{\partial w} + f_3(t) \frac{\partial}{\partial z};\tag{70}$$

- one more exotic operator associated with  $f_4$ :

$$v_6 = f_4(ne^{3\alpha t}) e^{-3\alpha t} \frac{\partial}{\partial n}; \quad (71)$$

- operator, which states, that we can freely choose arbitrary additive pressure at each moment of time, associated with  $f_5$ :

$$v_7 = f_5(t) \frac{\partial}{\partial p}; \quad (72)$$

- three time independent rotations:

$$v_8 = v \frac{\partial}{\partial w} - w \frac{\partial}{\partial v} + y \frac{\partial}{\partial z} - z \frac{\partial}{\partial y}; \quad (73)$$

$$v_9 = w \frac{\partial}{\partial u} - u \frac{\partial}{\partial w} + z \frac{\partial}{\partial x} - x \frac{\partial}{\partial z}; \quad (74)$$

$$v_{10} = u \frac{\partial}{\partial v} - v \frac{\partial}{\partial u} + x \frac{\partial}{\partial y} - y \frac{\partial}{\partial x}. \quad (75)$$

Now expression for variation of (13) is:

$$\int_V \left[ \delta n + n \left( \frac{\partial \delta u}{\partial u} + \frac{\partial \delta v}{\partial v} + \frac{\partial \delta w}{\partial w} \right) \right] dv_1 dv_2 dv_3 = 0, \quad (76)$$

Substitute (66-75) to (76) and get:

$$\int_V \left[ f_4(ne^{3\alpha t}) e^{-3\alpha t} - 12C_1 e^{\alpha t/2} n + n 3C_1 e^{\alpha t/2} \right] dv_1 dv_2 dv_3 = 0, \quad (77)$$

This means

$$C_1 = 0; \quad (78)$$

and

$$f_4 = 0. \quad (79)$$

Therefore symmetries of Fokker - Planck equation for incompressible fluid build the subgroup of (66-75) with excluded exotic symmetries  $v_1$  and  $v_6$ .

This list of symmetries is the same as the Navier - Stokes equations symmetries list (see ref. [7]), with exception of scaling symmetries. Our system has no scaling symmetries, because it contain additive frictional term.

## 7 Symmetries of linearized equations

Similar to previous section, we present only result of symmetries calculations without calculations details.

We use the same two-step process. As a result of the first step we get following expressions for variations of variables:

$$\delta n = C_1 n + \bar{n}, \quad (80)$$

$$\delta p = C_1 p + \bar{p}, \quad (81)$$

where  $C_1$  - arbitrary constant coefficient,  $\bar{n}$ ,  $\bar{p}$  - arbitrary solutions of linearized system (59-60) itself;

$$\delta t = C_2, \quad (82)$$

where  $C_2$  - one more arbitrary constant coefficient;

$$\delta x = r_2 z - r_3 y + C_3, \quad (83)$$

$$\delta y = r_3 x - r_1 z + C_4,$$

$$\delta z = r_1 y - r_2 x + C_5,$$

where  $r_1, r_2, r_3, C_3, C_4, C_5$  - arbitrary constant coefficients;

$$\delta u = r_2 w - r_3 v, \quad (84)$$

$$\delta v = r_3 u - r_1 w,$$

$$\delta w = r_1 v - r_2 u.$$

These expressions lead to the following expressions for symmetries operators:

- scaling operator associated with  $C_1$ :

$$v_1 = n \frac{\partial}{\partial n} + p \frac{\partial}{\partial p}; \quad (85)$$

- time shift operator associated with  $C_2$ :

$$v_2 = \frac{\partial}{\partial t}; \quad (86)$$

- three operators of shifts along  $x, y, z$  axes associated with  $C_3, C_4, C_5$  :

$$v_3 = \frac{\partial}{\partial x}; \quad (87)$$

$$v_4 = \frac{\partial}{\partial y}; \quad (88)$$

$$v_5 = \frac{\partial}{\partial z}; \quad (89)$$

- three rotations:

$$v_6 = v \frac{\partial}{\partial w} - w \frac{\partial}{\partial v} + y \frac{\partial}{\partial z} - z \frac{\partial}{\partial y}; \quad (90)$$

$$v_7 = w \frac{\partial}{\partial u} - u \frac{\partial}{\partial w} + z \frac{\partial}{\partial x} - x \frac{\partial}{\partial z}; \quad (91)$$

$$v_8 = u \frac{\partial}{\partial v} - v \frac{\partial}{\partial u} + x \frac{\partial}{\partial y} - y \frac{\partial}{\partial x}; \quad (92)$$

- infinite subgroup of linear system (59-60) solutions:

$$v_9 = \bar{n} \frac{\partial}{\partial n} + \bar{p} \frac{\partial}{\partial p}. \quad (93)$$

Now we go to second step. Expression for variation of (59) is:

$$\int_V \left[ \delta n + n \left( \frac{\partial \delta u}{\partial u} + \frac{\partial \delta v}{\partial v} + \frac{\partial \delta w}{\partial w} \right) \right] dv_1 dv_2 dv_3 = 0, \quad (94)$$

Substitute (80-84) to (94) and get:

$$\int_V (C_1 n) dv_1 dv_2 dv_3 = 0, \quad (95)$$

This means

$$C_1 = 0. \quad (96)$$

Therefore symmetries of linearized Fokker - Planck equation for incompressible fluid build the subgroup of (85-93) with excluded scaling symmetry  $v_1$ .

Let us give some small examples of invariant solutions.

Rotations (90-92) have following invariants:

$$r = \sqrt{x^2 + y^2 + z^2}; \quad (97)$$

$$U = \sqrt{u^2 + v^2 + w^2}; \quad (98)$$

$$U_r = xu + yv + zw. \quad (99)$$

According to PDE symmetries theory (ref. [7]), we should seek solutions of (59-60) of the following form:

$$n = n(t, r, U, U_r); \quad (100)$$

$$p = p(t, r); \quad (101)$$

(2) reads now

$$\begin{aligned} & \frac{\partial n}{\partial t} + \frac{U_r}{r} \frac{\partial n}{\partial r} + U^2 \frac{\partial n}{\partial U_r} - \alpha U \frac{\partial n}{\partial U} - \alpha U_r \frac{\partial n}{\partial U_r} - 3\alpha n + \\ & + \left(\frac{\alpha}{k}\right) \left(\frac{\alpha}{2\pi k}\right)^{3/2} \exp\left[-\frac{\alpha}{2k} U^2\right] \frac{U_r}{r} \frac{\partial p}{\partial r} = k \left( \frac{\partial^2 n}{\partial U^2} + \frac{2}{U} \frac{\partial n}{\partial U} + 2 \frac{U_r}{U} \frac{\partial^2 n}{\partial U_r \partial U} + r^2 \frac{\partial^2 n}{\partial U_r^2} \right). \end{aligned} \quad (102)$$

In this way we reduced (60) to equation with 4 independent variables. This equation is still not easy to solve. To give simpler example we drop  $U_r$  and keep only  $U$  variable:

$$n = n(t, r, U). \quad (103)$$

Then

$$\begin{aligned} & \frac{\partial n}{\partial t} + \frac{U_r}{r} \frac{\partial n}{\partial r} - \alpha U \frac{\partial n}{\partial U} - 3\alpha n + \\ & + \left(\frac{\alpha}{k}\right) \left(\frac{\alpha}{2\pi k}\right)^{3/2} \exp\left[-\frac{\alpha}{2k} U^2\right] \frac{U_r}{r} \frac{\partial p}{\partial r} = k \left( \frac{\partial^2 n}{\partial U^2} + \frac{2}{U} \frac{\partial n}{\partial U} \right). \end{aligned} \quad (104)$$

Equate coefficient by  $U_r$  to zero

$$\frac{\partial n}{\partial r} + \left(\frac{\alpha}{k}\right) \left(\frac{\alpha}{2\pi k}\right)^{3/2} \exp\left[-\frac{\alpha}{2k} U^2\right] \frac{\partial p}{\partial r} = 0. \quad (105)$$

The rest of (104) is

$$\frac{\partial n}{\partial t} - \alpha U \frac{\partial n}{\partial U} - 3\alpha n = k \left( \frac{\partial^2 n}{\partial U^2} + \frac{2}{U} \frac{\partial n}{\partial U} \right). \quad (106)$$

Integration of (106) gives

$$n + \left(\frac{\alpha}{k}\right) \left(\frac{\alpha}{2\pi k}\right)^{3/2} \exp\left[-\frac{\alpha}{2k} U^2\right] p = f(t, U). \quad (107)$$

Apply (59) to (107). We see, that  $p$  depends only on  $t$

$$p = g(t). \quad (108)$$

This means, that  $\frac{\partial n}{\partial r} = 0$  also and we need only solve (29) for  $n(t, U)$ .

## DISCUSSION

In this article we deducted the Fokker - Planck equation for incompressible fluid from the Gauss principle. This method can be useful for investigation of another cases of constrained movement. To promote this task we give three slightly different forms of equations for use of curvilinear coordinates.

Unfortunately, the actual solution of equations is a complicated task, because we move from the model with totally independent particles to model with interaction. This model is by necessity nonlinear. To make first steps to solution we use perturbation method. We construct linearized equations, but do not try to solve them at the moment - this is a task for another work.

It is of some interest to obtain symmetries group for our system. We find the group and compare it with symmetries of Navier - Stokes equations. One symmetry is missing in our case as a result of friction force presence.

We derive symmetries of linearized equations also and give some examples of invariant solutions. We see, that symmetries group of linearized system is not rich enough and is hardly usable to obtain physically interesting solutions.

## ACKNOWLEDGMENTS

We wish to thank Jos A. M. Vermaseren from NIKHEF (the Dutch Institute for Nuclear and High-Energy Physics), for he made his symbolic computations program FORM available for download for non-commercial purposes (see [8]).

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